

# Cohomology of $\mathfrak{osp}(1|2)$ acting on linear differential operators on the supercircle $S^1|1$

Imed Basdouri

Mabrouk Ben Ammar\*

February 5, 2008

## Abstract

We compute the first cohomology spaces  $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$  ( $\lambda, \mu \in \mathbb{R}$ ) of the Lie superalgebra  $\mathfrak{osp}(1|2)$  with coefficients in the superspace  $\mathfrak{D}_{\lambda, \mu}$  of linear differential operators acting on weighted densities on the supercircle  $S^1|1$ . The structure of these spaces was conjectured in [4]. In fact, we prove here that the situation is a little bit more complicated. (To appear in LMP.)

**Mathematics Subject Classification** (2000). 53D55

**Key words** : Cohomology, Orthosymplectic superalgebra.

## 1 Introduction

The space of weighted densities with weight  $\lambda$  (or  $\lambda$ -densities) on  $S^1$ , denoted by:

$$\mathcal{F}_\lambda = \{f(dx)^\lambda, f \in C^\infty(S^1)\}, \quad (\lambda \in \mathbb{R}),$$

is the space of sections of the line bundle  $(T^*S^1)^{\otimes \lambda}$ . Let  $\text{Vect}(S^1)$  be the Lie algebra of all vector fields  $F \frac{d}{dx}$  on  $S^1$ , ( $F \in C^\infty(S^1)$ ). With the *Lie derivative*,  $\mathcal{F}_\lambda$  is a  $\text{Vect}(S^1)$ -module. Alternatively, the  $\text{Vect}(S^1)$  action can be written as follows:

$$L_{F \frac{d}{dx}}^\lambda (f(dx)^\lambda) = (Ff' + \lambda fF')(dx)^\lambda, \quad (1.1)$$

where  $f', F'$  are  $\frac{df}{dx}, \frac{dF}{dx}$ .

Let  $A$  be a differential operator on  $S^1$ . We see  $A$  as the linear mapping  $f(dx)^\lambda \mapsto (Af)(dx)^\mu$  from  $\mathcal{F}_\lambda$  to  $\mathcal{F}_\mu$  ( $\lambda, \mu$  in  $\mathbb{R}$ ). Thus the space of differential operators is a  $\text{Vect}(S^1)$  module, denoted  $\mathcal{D}_{\lambda, \mu}$ . The  $\text{Vect}(S^1)$  action is:

$$L_X^{\lambda, \mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda. \quad (1.2)$$

---

\*Département de Mathématiques, Faculté des Sciences de Sfax, BP 802, 3038 Sfax, Tunisie.  
E.mails:basdourimed@yahoo.fr, mabrouk.benammar@fss.rnu.tn

If we restrict ourselves to the Lie subalgebra of  $\text{Vect}(S^1)$  generated by  $\left\{\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\right\}$ , isomorphic to  $\mathfrak{sl}(2)$ , we get a family of infinite dimensional  $\mathfrak{sl}(2)$  modules, still denoted  $\mathcal{D}_{\lambda,\mu}$ .

P. Lecomte, in [5], found the cohomology spaces  $H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$  and  $H^2(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$ . These spaces appear naturally in the problem of describing the deformations of the  $\mathfrak{sl}(2)$ -module  $\mathcal{D}$  of the differential operators acting on  $\mathcal{S}^n = \bigoplus_{k=-n}^n \mathcal{F}_{\frac{1+k}{2}}$ . More precisely, the first cohomology space  $H^1(\mathfrak{sl}(2); V)$  classifies the infinitesimal deformations of a  $\mathfrak{sl}(2)$  module  $V$  and the obstructions to integrability of a given infinitesimal deformation of  $V$  are elements of  $H^2(\mathfrak{sl}(2); V)$ . Thus, for instance, the infinitesimal deformations of the  $\mathfrak{sl}(2)$  module  $\mathcal{D}$  are classified by:

$$H^1(\mathfrak{sl}(2); \mathcal{D}) = \bigoplus_{k=0}^n H^1\left(\mathfrak{sl}(2); \mathcal{D}_{\frac{1-k}{2}, \frac{1+k}{2}}\right) \oplus \bigoplus_{k=-n}^n H^1\left(\mathfrak{sl}(2); \mathcal{D}_{\frac{1+k}{2}, \frac{1+k}{2}}\right).$$

In this paper we are interested to the study of the corresponding super structures. More precisely, we consider here the superspace  $S^{1|1}$  equipped with its standard *contact structure 1-form*  $\alpha$ , and introduce the superspace  $\mathfrak{F}_\lambda$  of  $\lambda$ -densities on the supercircle  $S^{1|1}$ .

Let  $\mathcal{K}(1)$  be the Lie superalgebra of contact vector fields,  $\mathfrak{F}_\lambda$  is naturally a  $\mathcal{K}(1)$ -module. For each  $\lambda, \mu$  in  $\mathbb{R}$ , any differential operator on  $S^{1|1}$  becomes a linear mapping from  $\mathfrak{F}_\lambda$  to  $\mathfrak{F}_\mu$ , thus the space of differential operators becomes a  $\mathcal{K}(1)$ -module denoted  $\mathfrak{D}_{\lambda,\mu}$ .

To the symplectic Lie algebra  $\mathfrak{sl}(2)$  corresponds the ortosymplectic Lie superalgebra  $\mathfrak{osp}(1|2)$  which is naturally realized as a subalgebra of  $\mathcal{K}(1)$ . Restricting our  $\mathcal{K}(1)$ -modules to  $\mathfrak{osp}(1|2)$ , we get  $\mathfrak{osp}(1|2)$ -modules still denoted  $\mathfrak{F}_\lambda, \mathfrak{D}_{\lambda,\mu}$ .

We compute here the first cohomology spaces  $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ , ( $\lambda, \mu$  in  $\mathbb{R}$ ), getting a result very close to the classical spaces  $H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$ . Especially, these spaces have the same dimension. Moreover, we give explicit formulae for all the non trivial 1-cocycles.

These spaces arise in the classification of infinitesimal deformations of the  $\mathfrak{osp}(1|2)$ -module of the differential operators acting on  $\mathcal{S}^n = \bigoplus_{k=1-n}^n \mathfrak{F}_{\frac{k}{2}}$ . We hope to be able to describe in the future all the deformations of this module.

## 2 Definitions and Notations

### 2.1 The Lie superalgebra of contact vector fields on $S^{1|1}$

We define the supercircle  $S^{1|1}$  through its space of functions,  $C^\infty(S^{1|1})$ . A  $C^\infty(S^{1|1})$  has the form:

$$F(x, \theta) = f_0(x) + \theta f_1(x),$$

where  $x$  is the even variable and  $\theta$  the odd variable: we have  $\theta^2 = 0$ . Even elements in  $C^\infty(S^{1|1})$  are the functions  $F(x, \theta) = f_0(x)$ , the functions  $F(x, \theta) = \theta f_1(x)$  are odd elements. Note  $p(F)$  the parity of a homogeneous function  $F$ .

Let  $\text{Vect}(S^{1|1})$  be the superspace of vector fields on  $S^{1|1}$ :

$$\text{Vect}(S^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_i \in C^\infty(S^{1|1}) \right\},$$

where  $\partial_\theta$  and  $\partial_x$  stand for  $\frac{\partial}{\partial\theta}$  and  $\frac{\partial}{\partial x}$ . The vector fields  $f(x)\partial_x$ , and  $\theta f(x)\partial_\theta$  are even, the vector fields  $\theta f(x)\partial_x$ , and  $f(x)\partial_\theta$  are odd. The superbracket of two vector fields is bilinear and defined for two homogeneous vector fields by:

$$[X, Y] = X \circ Y - (-1)^{p(X)p(Y)} Y \circ X.$$

Denote  $\mathfrak{L}_X$  the Lie derivative of a vector field, acting on the space of functions, forms, vector fields,...

The supercircle  $S^{1|1}$  is equipped with the standard contact structure given by the following even 1-form:

$$\alpha = dx + \theta d\theta.$$

We consider the Lie superalgebra  $\mathcal{K}(1)$  of contact vector fields on  $S^{1|1}$ . That is,  $\mathcal{K}(1)$  is the superspace of conformal vector fields on  $S^{1|1}$  with respect to the 1-form  $\alpha$ :

$$\mathcal{K}(1) = \{X \in \text{Vect}(S^{1|1}) \mid \text{there exists } F \in C^\infty(S^{1|1}) \text{ such that } \mathfrak{L}_X(\alpha) = F\alpha\}.$$

Let us define the vector fields  $\eta$  and  $\bar{\eta}$  by

$$\eta = \partial_\theta + \theta \partial_x, \quad \bar{\eta} = \partial_\theta - \theta \partial_x.$$

Then any contact vector field on  $S^{1|1}$  can be written in the following explicit form:

$$X_F = F\partial_x + \frac{1}{2}\eta(F)(\partial_\theta - \theta\partial_x) = -F\bar{\eta}^2 + \frac{1}{2}\eta(F)\bar{\eta}, \quad \text{where } F \in C^\infty(S^{1|1}).$$

Of course,  $\mathcal{K}(1)$  is a subalgebra of  $\text{Vect}(S^{1|1})$ , and  $\mathcal{K}(1)$  acts on  $C^\infty(S^{1|1})$  through:

$$\mathfrak{L}_{X_F}(G) = FG' + \frac{1}{2}(-1)^{(p(F)+1)p(G)}\bar{\eta}(F) \cdot \bar{\eta}(G). \quad (2.3)$$

Let us define the contact bracket on  $C^\infty(S^{1|1})$  as the bilinear mapping such that, for a couple of homogenous functions  $F, G$ ,

$$\{F, G\} = FG' - F'G + \frac{1}{2}(-1)^{p(F)+1}\bar{\eta}(F) \cdot \bar{\eta}(G), \quad (2.4)$$

Then the bracket of  $\mathcal{K}(1)$  can be written as:

$$[X_F, X_G] = X_{\{F, G\}}.$$

## 2.2 The superalgebra $\mathfrak{osp}(1|2)$

Recall the Lie algebra  $\mathfrak{sl}(2)$  is isomorphic to the Lie subalgebra of  $\text{Vect}(S^1)$  generated by

$$\left\{ \frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx} \right\}.$$

Similarly, we now consider the orthosymplectic Lie superalgebra as a subalgebra of  $\mathcal{K}(1)$ :

$$\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta}, X_\theta).$$

The space of even elements is isomorphic to  $\mathfrak{sl}(2)$ :

$$(\mathfrak{osp}(1|2))_0 = \text{Span}(X_1, X_x, X_{x^2}) = \mathfrak{sl}(2).$$

The space of odd elements is two dimensional:

$$(\mathfrak{osp}(1|2))_1 = \text{Span}(X_{x\theta}, X_\theta).$$

The new commutation relations are

$$\begin{aligned} [X_{x^2}, X_\theta] &= -X_{x\theta}, & [X_x, X_\theta] &= -\frac{1}{2}X_\theta, & [X_1, X_\theta] &= 0, \\ [X_{x^2}, X_{x\theta}] &= 0, & [X_x, X_{x\theta}] &= \frac{1}{2}X_{x\theta}, & [X_1, X_{x\theta}] &= X_\theta, \\ [X_{x\theta}, X_\theta] &= \frac{1}{2}X_x. \end{aligned}$$

### 2.3 The space of weighted densities on $S^{1|1}$

In the super setting, by replacing  $dx$  by the 1-form  $\alpha$ , we get analogous definition for weighted densities i.e. we define the space of  $\lambda$ -densities as

$$\mathfrak{F}_\lambda = \{ \phi = F(x, \theta) \alpha^\lambda \mid F(x, \theta) \in C^\infty(S^{1|1}) \}. \quad (2.5)$$

As a vector space,  $\mathfrak{F}_\lambda$  is isomorphic to  $C^\infty(S^{1|1})$ , but the Lie derivative of the density  $G\alpha^\lambda$  along the vector field  $X_F$  in  $\mathcal{K}(1)$  is now:

$$\mathfrak{L}_{X_F}(G\alpha^\lambda) = \mathfrak{L}_{X_F}^\lambda(G)\alpha^\lambda, \quad \text{with} \quad \mathfrak{L}_{X_F}^\lambda(G) = \mathfrak{L}_{X_F}(G) + \lambda F'G. \quad (2.6)$$

Or, if we put  $F = a(x) + b(x)\theta$ ,  $G = g_0(x) + g_1(x)\theta$ ,

$$\mathfrak{L}_{X_F}^\lambda(G) = L_{a\partial_x}^\lambda(g_0) + \frac{1}{2}bg_1 + \left( L_{a\partial_x}^{\lambda+\frac{1}{2}}(g_1) + \lambda g_0 b' + \frac{1}{2}g_0' b \right) \theta. \quad (2.7)$$

Especially, we have

$$\begin{cases} \mathfrak{L}_{X_a}^\lambda(g_0) = L_{a\partial_x}^\lambda(g_0), & \mathfrak{L}_{X_a}^\lambda(g_1\theta) = \theta L_{a\partial_x}^{\lambda+\frac{1}{2}}(g_1), \\ \mathfrak{L}_{X_{b\theta}}^\lambda(g_0) = (\lambda g_0 b' + \frac{1}{2}g_0' b)\theta & \text{and} \\ \mathfrak{L}_{X_{b\theta}}^\lambda(g_1\theta) = \frac{1}{2}bg_1. \end{cases}$$

Of course, for all  $\lambda$ ,  $\mathfrak{F}_\lambda$  is a  $\mathcal{K}(1)$ -module:

$$[\mathfrak{L}_{X_F}^\lambda, \mathfrak{L}_{X_G}^\lambda] = \mathfrak{L}_{[X_F, X_G]}^\lambda.$$

We thus obtain a one-parameter family of  $\mathcal{K}(1)$ -modules on  $C^\infty(S^{1|1})$  still denoted by  $\mathfrak{F}_\lambda$ .

## 2.4 Differential Operators on Weighted Densities

A differential operator on  $S^{1|1}$  is an operator on  $C^\infty(S^{1|1})$  of the following form:

$$A = \sum_{i=0}^{\ell} \tilde{a}_i(x, \theta) \partial_x^i + \sum_{i=0}^{\ell} \tilde{b}_i(x, \theta) \partial_x^i \partial_\theta.$$

In [4], it is proved that any local operator  $A$  on  $S^{1|1}$  is in fact a differential operator.

Of course, any differential operator defines a linear mapping from  $\mathfrak{F}_\lambda$  to  $\mathfrak{F}_\mu$  for any  $\lambda, \mu \in \mathbb{R}$ , thus the space of differential operators becomes a family of  $\mathcal{K}(1)$  and  $\mathfrak{osp}(1|2)$  modules denoted  $\mathfrak{D}_{\lambda, \mu}$ , for the natural action:

$$\mathfrak{L}_{X_F}^{\lambda, \mu}(A) = \mathfrak{L}_{X_F}^\mu \circ A - (-1)^{p(A)p(F)} A \circ \mathfrak{L}_{X_F}^\lambda. \quad (2.8)$$

## 3 The space $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu})$

### 3.1 Lie superalgebra cohomology (see [2])

Let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be a Lie superalgebra and  $A = A_0 \oplus A_1$  a  $\mathfrak{g}$  module. We define the *cochain complex* associated to the module as an exact sequence:

$$0 \longrightarrow C^0(\mathfrak{g}, A) \longrightarrow \cdots \longrightarrow C^{q-1}(\mathfrak{g}, A) \xrightarrow{\delta^{q-1}} C^q(\mathfrak{g}, A) \cdots$$

The spaces  $C^q(\mathfrak{g}, A)$  are the spaces of super skew-symmetric  $q$  linear mappings:

$$C^0(\mathfrak{g}, A) = A, \quad C^q(\mathfrak{g}, A) = \bigoplus_{q_0+q_1=q} \text{Hom}\left(\bigwedge^{q_0} \mathfrak{g}_0 \otimes S^{q_1} \mathfrak{g}_1, A\right).$$

Elements of  $C^q(\mathfrak{g}, A)$  are called *cochains*. The spaces  $C^q(\mathfrak{g}, A)$  is  $\mathbb{Z}_2$  graded:

$$C^q(\mathfrak{g}, A) = C_0^q(\mathfrak{g}, A) + C_1^q(\mathfrak{g}, A), \quad \text{with} \quad C_p^q(\mathfrak{g}, A) = \bigoplus_{\substack{q_0+q_1=q \\ q_1+r=p \pmod{2}}} \text{Hom}\left(\bigwedge^{q_0} \mathfrak{g}_0 \otimes S^{q_1} \mathfrak{g}_1, A_r\right).$$

The linear mapping  $\delta^q$  (or, briefly  $\delta$ ) is called the *coboundary operator*. This operator is a generalization of the usual Chevalley coboundary operator for Lie algebra to the case of Lie superalgebra. Explicitly, it is defined as follows. Take a cochain  $c \in C^q(\mathfrak{g}, A)$ , then for  $q_0, q_1$  with  $q_0 + q_1 = q + 1$ ,  $\delta^q c$  is:

$$\begin{aligned}
& \delta^q c(g_1, \dots, g_{q_0}, h_1, \dots, h_{q_1}) \\
&= \sum_{1 \leq s < t \leq q_0} (-1)^{s+t-1} c([g_s, g_t], g_1, \dots, \hat{g}_s, \dots, \hat{g}_t, \dots, g_{q_0}, h_1, \dots, h_{q_1}) \\
&\quad + \sum_{s=1}^{q_0} \sum_{t=1}^{q_1} (-1)^{s-1} c(g_1, \dots, \hat{g}_s, \dots, g_{q_0}, [g_s, h_t], h_1, \dots, \hat{h}_t, \dots, h_{q_1}) \\
&\quad + \sum_{1 \leq s < t \leq q_1} c([h_s, h_t], g_1, \dots, g_{q_0}, h_1, \dots, \hat{h}_s, \dots, \hat{h}_t, \dots, h_{q_1}) \\
&\quad + \sum_{s=1}^{q_0} (-1)^s g_s c(g_1, \dots, \hat{g}_s, \dots, g_{q_0}, h_1, \dots, h_{q_1}) \\
&\quad + (-1)^{q_0-1} \sum_{s=1}^{q_1} h_s c(g_1, \dots, g_{q_0}, h_1, \dots, \hat{h}_s, \dots, h_{q_1}).
\end{aligned}$$

where  $g_1, \dots, g_{q_0}$  are in  $\mathfrak{g}_0$  and  $h_1, \dots, h_{q_1}$  in  $\mathfrak{g}_1$ .

The relation  $\delta^q \circ \delta^{q-1} = 0$  holds. The kernel of  $\delta^q$ , denoted  $Z^q(\mathfrak{g}, A)$ , is the space of  $q$  cocycles, among them, the elements in the range of  $\delta^{q-1}$  are called  $q$  coboundaries. We note  $B^q(\mathfrak{g}, A)$  the space of  $q$  coboundaries.

By definition, the  $q^{th}$  cohomolgy space is the quotient space

$$H^q(\mathfrak{g}, A) = Z^q(\mathfrak{g}, A) / B^q(\mathfrak{g}, A).$$

One can check that  $\delta^q(C_p^q(\mathfrak{g}, A)) \subset C_p^{q+1}(\mathfrak{g}, A)$  and then we get the following sequences

$$0 \longrightarrow C_p^0(\mathfrak{g}, A) \longrightarrow \dots \longrightarrow C_p^{q-1}(\mathfrak{g}, A) \xrightarrow{\delta^{q-1}} C_p^q(\mathfrak{g}, A) \dots,$$

where  $p = 0$  or  $1$ . The cohomology spaces are thus graded by

$$H_p^q(\mathfrak{g}, A) = \text{Ker} \delta^q|_{C_p^q(\mathfrak{g}, A)} / \delta^{q-1}(C_p^{q-1}(\mathfrak{g}, A)).$$

### 3.2 The main theorem

The main result in this paper is the following:

**Theorem 3.1.** *The cohomolgy spaces  $H_p^1(\mathfrak{g}, \mathfrak{D}_{\lambda, \mu})$  are finite dimensional. An explicit description of these spaces is the following:*

1) The space  $H_0^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$  is

$$H_0^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \quad (3.9)$$

A base for the space  $H_0^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\lambda})$  is given by the cohomology class of the 1-cocycle:

$$\Upsilon_{\lambda,\lambda}(X_F) = F'. \quad (3.10)$$

2) The space  $H_1^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$  is

$$H_1^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \lambda = \frac{1-k}{2}, \mu = \frac{k}{2}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.11)$$

A base for the space  $H_1^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{1-k}{2}, \frac{k}{2}})$  is given by the cohomology classes of the 1-cocycles:

$$\begin{aligned} \Upsilon_{\frac{1-k}{2}, \frac{k}{2}}(X_F) &= \bar{\eta}^2(F) \bar{\eta}^{2k-1}, \\ \tilde{\Upsilon}_{\frac{1-k}{2}, \frac{k}{2}}(X_F) &= (k-1)\eta^4(F) \bar{\eta}^{2k-3} + \eta^3(F) \bar{\eta}^{2k-2}. \end{aligned} \quad (3.12)$$

Note that the 1-cocycle  $\tilde{\Upsilon}_{\frac{1-k}{2}, \frac{k}{2}}$  coincides with the 1-cocycle  $\gamma_{2k-1}$  given by Gargoubi et al. in [4]. The proof of Theorem 3.1 will be the subject of subsection 3.4.

### 3.3 Relationship between $H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$ and $H^1(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$

Before proving the theorem 3.1 we present here some results illustrating the analogy between the cohomology spaces in super and classical settings.

First, note that:

- 1) As a  $\mathfrak{sl}(2)$ -module, we have  $\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$  and  $\mathfrak{osp}(1|2) \simeq \mathfrak{sl}(2) \oplus \Pi(\mathfrak{h})$ , where  $\mathfrak{h}$  is the subspace of  $\mathcal{F}_{-\frac{1}{2}}$  spanned by  $\{dx^{-\frac{1}{2}}, xdx^{-\frac{1}{2}}\}$  and  $\Pi$  is the change of parity.
- 2) As a  $\mathfrak{sl}(2)$ -module, we have for the homogeneous components of  $\mathfrak{D}_{\lambda,\mu}$ :

$$(\mathfrak{D}_{\lambda,\mu})_0 \simeq \mathcal{D}_{\lambda,\mu} \oplus \mathcal{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}} \quad \text{and} \quad (\mathfrak{D}_{\lambda,\mu})_1 \simeq \Pi(\mathcal{D}_{\lambda+\frac{1}{2}, \mu} \oplus \mathcal{D}_{\lambda, \mu+\frac{1}{2}}).$$

**Proposition 3.1.** Any 1-cocycle  $\Upsilon \in Z^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ , is decomposed into  $(\Upsilon', \Upsilon'')$  in  $Hom(\mathfrak{sl}(2); \mathfrak{D}_{\lambda,\mu}) \oplus Hom(\mathfrak{h}; \mathfrak{D}_{\lambda,\mu})$ .  $\Upsilon'$  and  $\Upsilon''$  are solutions of the following equations:

$$\Upsilon'([X_{g_1}, X_{g_2}]) - \mathfrak{L}_{X_{g_1}}^{\lambda,\mu} \Upsilon'(X_{g_2}) + \mathfrak{L}_{X_{g_2}}^{\lambda,\mu} \Upsilon'(X_{g_1}) = 0, \quad (3.13)$$

$$\Upsilon''([X_g, X_{h\theta}]) - \mathfrak{L}_{X_g}^{\lambda,\mu} \Upsilon''(X_{h\theta}) + \mathfrak{L}_{X_{h\theta}}^{\lambda,\mu} \Upsilon'(X_g) = 0, \quad (3.14)$$

$$\Upsilon'([X_{h_1\theta}, X_{h_2\theta}]) - \mathfrak{L}_{X_{h_1\theta}}^{\lambda,\mu} \Upsilon''(X_{h_2\theta}) - \mathfrak{L}_{X_{h_2\theta}}^{\lambda,\mu} \Upsilon''(X_{h_1\theta}) = 0, \quad (3.15)$$

here,  $g, g_1, g_2$  are polynomials in the variable  $x$ , with degree at most 2, and  $h, h_1, h_2$  are affine functions in the variable  $x$ .

Proof. The equations (3.13), (3.14) and (3.15) are equivalent to the fact that  $\Upsilon$  is a 1-cocycle. For any  $X_F, X_G \in \mathfrak{osp}(1|2)$ ,

$$\delta\Upsilon(X_F, X_G) := \Upsilon([X_F, X_G]) - \mathfrak{L}_{X_F}^{\lambda, \mu} \Upsilon(X_G) + (-1)^{p(F)p(G)} \mathfrak{L}_{X_G}^{\lambda, \mu} \Upsilon(X_F) = 0.$$

□

According to the  $\mathbb{Z}_2$ -grading, the even component  $\Upsilon_0$  and the odd component  $\Upsilon_1$  of any 1-cocycle  $\Upsilon$  can be decomposed as  $\Upsilon_0 = (\Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{110}, \Upsilon_{11\frac{1}{2}})$  and  $\Upsilon_1 = (\Upsilon_{010}, \Upsilon_{01\frac{1}{2}}, \Upsilon_{100}, \Upsilon_{10\frac{1}{2}})$ , where

$$\left\{ \begin{array}{lll} \Upsilon_{000} : & \mathfrak{sl}(2) & \rightarrow \mathcal{D}_{\lambda, \mu}, \\ \Upsilon_{00\frac{1}{2}} : & \mathfrak{sl}(2) & \rightarrow \mathcal{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}, \\ \Upsilon_{110} : & \mathfrak{h} & \rightarrow \mathcal{D}_{\lambda, \mu+\frac{1}{2}}, \\ \Upsilon_{11\frac{1}{2}} : & \mathfrak{h} & \rightarrow \mathcal{D}_{\lambda+\frac{1}{2}, \mu} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lll} \Upsilon_{010} : & \mathfrak{sl}(2) & \rightarrow \mathcal{D}_{\lambda, \mu+\frac{1}{2}}, \\ \Upsilon_{01\frac{1}{2}} : & \mathfrak{sl}(2) & \rightarrow \mathcal{D}_{\lambda+\frac{1}{2}, \mu}, \\ \Upsilon_{100} : & \mathfrak{h} & \rightarrow \mathcal{D}_{\lambda, \mu}, \\ \Upsilon_{10\frac{1}{2}} : & \mathfrak{h} & \rightarrow \mathcal{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}. \end{array} \right.$$

The decomposition  $\Upsilon = (\Upsilon', \Upsilon'')$  given in proposition 3.1 corresponds to

$$\Upsilon' = (\Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{010}, \Upsilon_{01\frac{1}{2}}) \quad \text{and} \quad \Upsilon'' = (\Upsilon_{110}, \Upsilon_{11\frac{1}{2}}, \Upsilon_{100}, \Upsilon_{10\frac{1}{2}}).$$

By considering the equation (3.13), we can see the components  $\Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{010}$  and  $\Upsilon_{01\frac{1}{2}}$  as 1-cocycles on  $\mathfrak{sl}(2)$  with coefficients respectively in  $\mathcal{D}_{\lambda, \mu}, \mathcal{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}, \mathcal{D}_{\lambda, \mu+\frac{1}{2}},$  and  $\mathcal{D}_{\lambda+\frac{1}{2}, \mu}$ .

The first cohomology space  $H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda, \mu})$  was computed by Gargoubi and Lecomte [3, 5]. The result is the following:

$$H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R} & \text{if } \lambda = \mu \\ \mathbb{R}^2 & \text{if } (\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \text{ where } k \in \mathbb{N} \setminus \{0\} \\ 0 & \text{otherwise.} \end{cases} \quad (3.16)$$

The space  $H^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda, \lambda})$  is generated by the cohomology class of the 1-cocycle

$$C'_\lambda(F \frac{d}{dx})(f dx^\lambda) = F' f dx^\lambda. \quad (3.17)$$

For  $k \in \mathbb{N} \setminus \{0\}$ , the space  $H^1(\mathfrak{sl}(2); \mathcal{D}_{\frac{1-k}{2}, \frac{1+k}{2}})$  is generated by the cohomology classes of the 1-cocycles,  $C_k$  and  $\tilde{C}_k$  defined by

$$C_k(F \frac{d}{dx})(f dx^{\frac{1-k}{2}}) = F' f^{(k)} dx^{\frac{1+k}{2}} \quad \text{and} \quad \tilde{C}_k(F \frac{d}{dx})(f dx^{\frac{1-k}{2}}) = F'' f^{(k-1)} dx^{\frac{1+k}{2}}. \quad (3.18)$$

We shall need the following description of  $\mathfrak{sl}(2)$  invariant mappings.

**Lemma 3.2.** *Let*

$$A : \mathfrak{h} \times \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu, \quad (h dx^{-\frac{1}{2}}, f dx^\lambda) \mapsto A(h, f) dx^\mu$$

be a bilinear differential operator. If  $A$  is  $\mathfrak{sl}(2)$ -invariant then

$$\mu = \lambda - \frac{1}{2} + k, \quad \text{where } k \in \mathbb{N}$$

and the following relation holds

$$A_k(h, f) = a_k(hf^{(k)} + k(2\lambda + k - 1)h'f^{(k-1)}), \quad \text{where } k(k-1)(2\lambda + k - 1)(2\lambda + k - 2)a_k = 0.$$

Proof. A straightforward computation.  $\square$

Now, let us study the relationship between these 1-cocycles and their analogues in the super setting. We know that any element  $\Upsilon \in Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$  is decomposed into  $\Upsilon = \Upsilon' + \Upsilon''$  where  $\Upsilon' \in \text{Hom}(\mathfrak{sl}(2), \mathfrak{D}_{\lambda, \mu})$  and  $\Upsilon'' \in \text{Hom}(\mathfrak{h}, \mathfrak{D}_{\lambda, \mu})$ . The following lemma shows the close relationship between the cohomology spaces  $H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$  and  $H^1(\mathfrak{sl}(2), \mathfrak{D}_{\lambda, \mu})$ .

**Lemma 3.3.** *The 1-cocycle  $\Upsilon$  is a coboundary for  $\mathfrak{osp}(1|2)$  if and only if  $\Upsilon'$  is a coboundary for  $\mathfrak{sl}(2)$ .*

Proof. It is easy to see that if  $\Upsilon$  is a coboundary for  $\mathfrak{osp}(1|2)$  then  $\Upsilon'$  is a coboundary over  $\mathfrak{sl}(2)$ . Now, assume that  $\Upsilon'$  is a coboundary for  $\mathfrak{sl}(2)$ , that is, there exists  $\tilde{A} \in \mathfrak{D}_{\lambda, \mu}$  such that for all  $g$  polynomial in the variable  $x$  with degree at most 2

$$\Upsilon'(X_g) = \mathfrak{L}_{X_g}^{\lambda, \mu} \tilde{A}.$$

By replacing  $\Upsilon$  by  $\Upsilon - \delta \tilde{A}$ , we can suppose that  $\Upsilon' = 0$ . But, in this case, the map  $\Upsilon''$  must satisfy, for all  $h, h_1, h_2$  polynomial with degree 0 or 1 and  $g$  polynomial with degree 0, 1 or 2, the following equations

$$\mathfrak{L}_{X_g}^{\lambda, \mu} \Upsilon''(X_{h\theta}) - \Upsilon''([X_g, X_{h\theta}]) = 0, \quad (3.19)$$

$$\mathfrak{L}_{X_{h_1\theta}}^{\lambda, \mu} \Upsilon''(X_{h_2\theta}) + \mathfrak{L}_{X_{h_2\theta}}^{\lambda, \mu} \Upsilon''(X_{h_1\theta}) = 0. \quad (3.20)$$

1) If  $\Upsilon$  is an even 1-cocycle then  $\Upsilon''$  is decomposed into  $\Upsilon''_{00} : \mathfrak{h} \otimes \mathcal{F}_{\lambda + \frac{1}{2}} \rightarrow \mathcal{F}_\mu$  and  $\Upsilon''_{01} : \mathfrak{h} \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_{\mu + \frac{1}{2}}$ . The equation (3.19) tell us that  $\Upsilon''_{00}$  and  $\Upsilon''_{01}$  are  $\mathfrak{sl}(2)$  invariant bilinear maps. Therefore, the expressions of  $\Upsilon''_{00}$  and  $\Upsilon''_{01}$  are given by Lemma 3.2. So, we must have  $\mu = \lambda + k = (\lambda + \frac{1}{2}) - \frac{1}{2} + k$  (and then  $\mu + \frac{1}{2} = \lambda - \frac{1}{2} + k + 1$ ). More precisely, using the equation (3.20), we get up to a factor:

$$\Upsilon = \begin{cases} 0 & \text{if } k(k-1)(2\lambda + k)(2\lambda + k - 1) \neq 0 \text{ or } k = 1 \text{ and } \lambda \notin \{0, -\frac{1}{2}\}, \\ \delta(\theta \partial_\theta \partial_x^k) & \text{if } (\lambda, \mu) = (\frac{-k}{2}, \frac{k}{2}), \\ \delta(\partial_x^k - \theta \partial_\theta \partial_x^k) & \text{if } (\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \text{ or } \lambda = \mu. \end{cases}$$

2) If  $\Upsilon$  is an odd 1-cocycle then  $\Upsilon''$  is decomposed into  $\Upsilon''_{00} : \mathfrak{h} \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$  and  $\Upsilon''_{01} : \mathfrak{h} \otimes \mathcal{F}_{\lambda+\frac{1}{2}} \rightarrow \mathcal{F}_{\mu+\frac{1}{2}}$ . As in the previous case, the expressions of  $\Upsilon''_{00}$  and  $\Upsilon''_{01}$  are given by Lemma 3.2. So, we must have  $\mu = \lambda - \frac{1}{2} + k$  (and then  $\mu + \frac{1}{2} = (\lambda + \frac{1}{2}) - \frac{1}{2} + k$ .) More precisely, using the equation (3.20), we get:

$$\Upsilon = \begin{cases} 0 & \text{if } k(k-1)(2\lambda+k-1) \neq 0 \\ \delta(\theta) & \text{if } \mu = \lambda - \frac{1}{2}, \\ \delta(\partial_\theta) & \text{if } \mu = \lambda + \frac{1}{2}, \\ \delta(\theta\partial_x^k) & \text{if } (\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2}). \end{cases}$$

□

Now, the space  $Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$  of 1-cocycles is  $\mathbb{Z}_2$ -graded:

$$Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}) = Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})_0 \oplus Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})_1. \quad (3.21)$$

Therefore, any element  $\Upsilon \in Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$  is decomposed into an even part  $\Upsilon_0$  and odd part  $\Upsilon_1$ . Each of  $\Upsilon_0$  and  $\Upsilon_1$  is decomposed into two components:  $\Upsilon_0 = (\Upsilon_{00}, \Upsilon_{11})$  and  $\Upsilon_1 = (\Upsilon_{01}, \Upsilon_{10})$ , where

$$\begin{cases} \Upsilon_{00} : \mathfrak{sl}(2) \rightarrow (\mathfrak{D}_{\lambda,\mu})_0, \\ \Upsilon_{11} : \mathfrak{h} \rightarrow (\mathfrak{D}_{\lambda,\mu})_1, \end{cases} \quad \text{and} \quad \begin{cases} \Upsilon_{01} : \mathfrak{sl}(2) \rightarrow (\mathfrak{D}_{\lambda,\mu})_1, \\ \Upsilon_{10} : \mathfrak{h} \rightarrow (\mathfrak{D}_{\lambda,\mu})_0. \end{cases}$$

The components  $\Upsilon_{11}$  and  $\Upsilon_{10}$  of  $\Upsilon_0$  and  $\Upsilon_1$  are also decomposed as follows:  $\Upsilon_{11} = \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}$  and  $\Upsilon_{10} = \Upsilon_{100} + \Upsilon_{10\frac{1}{2}}$ , where  $\Upsilon_{110} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda,\mu+\frac{1}{2}})$ ,  $\Upsilon_{11\frac{1}{2}} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda+\frac{1}{2},\mu})$ ,  $\Upsilon_{100} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda,\mu})$ ,  $\Upsilon_{10\frac{1}{2}} \in \text{Hom}(\mathfrak{h}, \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}})$ .

As in [1], the following lemma gives the general form of each of  $\Upsilon_{110}$  and  $\Upsilon_{11\frac{1}{2}}$ .

**Lemma 3.4.** *Up to a coboundary, the maps  $\Upsilon_{110}$ ,  $\Upsilon_{11\frac{1}{2}}$ ,  $\Upsilon_{100}$  and  $\Upsilon_{10\frac{1}{2}}$  are given by*

$$\begin{aligned} \Upsilon_{110}(X_{h\theta}) &= a_0 h \theta \partial_x^k + a_1 h' \theta \partial_x^{k-1} \quad \text{and} \quad \Upsilon_{11\frac{1}{2}}(X_{h\theta}) = b_0 h \partial_\theta \partial_x^k + b_1 h' \partial_\theta \partial_x^{k-1}, \\ \Upsilon_{110}(X_{h\theta}) &= c_0 h \theta \partial_x^k + c_1 h' \theta \partial_x^{k-1} \quad \text{and} \quad \Upsilon_{11\frac{1}{2}}(X_{h\theta}) = d_0 h \partial_\theta \partial_x^k + d_1 h' \partial_\theta \partial_x^{k-1}, \end{aligned}$$

where the coefficients  $a_i$ ,  $b_i$ ,  $c_i$ , and  $d_i$  are constants.

*Proof.* The coefficients  $a_i$ ,  $b_i$ ,  $c_i$ , and  $d_i$  a priori are some functions of  $x$ , but we shall now prove  $\partial_x a_i = \partial_x b_i = 0$  (and similarly  $\partial_x c_i = \partial_x d_i = 0$ ). To do that, we shall simply show that  $\mathfrak{L}_{\partial_x}^{\lambda,\mu}(\Upsilon_{11}) = 0$ .

First, for all  $h$  polynomial with degree 0 or 1, we have

$$(\mathfrak{L}_{\partial_x}^{\lambda,\mu} \Upsilon_{11})(X_{h\theta}) = \mathfrak{L}_{\partial_x}^{\lambda,\mu}(\Upsilon_{11}(X_{h\theta})) - \Upsilon_{11}([\partial_x, X_{h\theta}]). \quad (3.22)$$

On the other hand, from Lemma 3.3, it follows that, up to a coboundary,  $\Upsilon_{00}$  is a linear combination of some 1-cocycles for  $\mathfrak{sl}(2)$  given by (3.17) and (3.18). So, we have  $\Upsilon_{00}(\partial_x) = 0$  and then

$$\mathfrak{L}_{X_{h\theta}}^{\lambda,\mu}(\Upsilon_{00}(\partial_x)) = 0.$$

Therefore, the equation (3.22) becomes, for all  $h$ ,

$$-(\mathfrak{L}_{\partial_x}^{\lambda,\mu}\Upsilon_{11})(X_{h\theta}) = \Upsilon_{11}([\partial_x, X_{h\theta}]) - \mathfrak{L}_{\partial_x}^{\lambda,\mu}(\Upsilon_{11}(X_{h\theta})) + \mathfrak{L}_{X_{h\theta}}^{\lambda,\mu}(\Upsilon_{00}(\partial_x)). \quad (3.23)$$

The right-hand side of (3.23) is nothing but  $\delta\Upsilon_0(\partial_x, X_{h\theta})$ . But,  $\Upsilon_0$  is a 1-cocycle, then  $\mathfrak{L}_{\partial_x}^{\lambda,\mu}(\Upsilon_{11}) = 0$ . Lemma 3.4 is proved.  $\square$

### 3.4 Proof of Theorem 3.1

The first cohomology space  $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$  inherits the  $\mathbb{Z}_2$ -grading from (3.21) and is decomposed into odd and an even subspaces:

$$H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}) = H_0^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}) \oplus H_1^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}).$$

We compute each part separately.

1) Let  $\Upsilon_0$  be a non trivial even 1-cocycle for  $\mathfrak{osp}(1|2)$  in  $\mathfrak{D}_{\lambda,\mu}$ . According to the  $\mathbb{Z}_2$ -grading,  $\Upsilon_0$  should retain the following general form:  $\Upsilon_0 = \Upsilon_{000} + \Upsilon_{00\frac{1}{2}} + \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}$  such that

$$\begin{cases} \Upsilon_{000} & : \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda,\mu}, \\ \Upsilon_{00\frac{1}{2}} & : \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda\frac{1}{2}f, \mu+\frac{1}{2}}, \\ \Upsilon_{110} & : \mathfrak{h} \rightarrow \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\ \Upsilon_{11\frac{1}{2}} & : \mathfrak{h} \rightarrow \mathcal{D}_{\lambda+\frac{1}{2}, \mu}. \end{cases} \quad (3.24)$$

Then, by using Lemma 3.3, we deduce that, up to coboundary,  $\Upsilon_{000}$  and  $\Upsilon_{00\frac{1}{2}}$  can be expressed in terms of  $C'_\lambda$ ,  $C_k$  and  $\tilde{C}_k$  where  $\lambda \in \mathbb{R}$  and  $k \in \mathbb{N} \setminus \{0\}$ . We thus consider three cases:

- i)  $\lambda = \mu$ ,  $\Upsilon_{000} = \alpha C'_\lambda$ , and  $\Upsilon_{00\frac{1}{2}} = \beta C'_{\lambda\frac{1}{2}f}$ .
- ii)  $(\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2})$ ,  $\Upsilon_{000} = \alpha_1 C_k + \alpha_2 \tilde{C}_k$ , and  $\Upsilon_{00\frac{1}{2}} = 0$ .
- iii)  $(\lambda, \mu) = (\frac{-k}{2}, \frac{k}{2})$ ,  $\Upsilon_{000} = 0$  and  $\Upsilon_{00\frac{1}{2}} = \alpha_1 C_k + \alpha_2 \tilde{C}_k$ .

Put  $\Upsilon' = \Upsilon_{000} + \Upsilon_{00\frac{1}{2}}$  and  $\Upsilon'' = \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}$ . In each case, the 1-cocycle  $\Upsilon_0$  must satisfy

$$\begin{cases} \Upsilon''[X_g, X_{\theta h}] & = \mathfrak{L}_{X_g}^{\lambda,\mu}\Upsilon''(X_{\theta h}) - \mathfrak{L}_{X_{\theta h}}^{\lambda,\mu}\Upsilon'(X_g), \\ \Upsilon'[X_{\theta h_1}, X_{\theta h_2}] & = \mathfrak{L}_{X_{\theta h_1}}^{\lambda,\mu}\Upsilon''(X_{\theta h_2}) + \mathfrak{L}_{X_{\theta h_2}}^{\lambda,\mu}\Upsilon''(X_{\theta h_1}), \end{cases} \quad (3.25)$$

where  $h$ ,  $h_1$ , and  $h_2$  are polynomials of degree 0 or 1,  $g$  polynomial of degree 0, 1 or 2.

Now, thanks to Lemma 3.4, we can write

$$\Upsilon_{110}(X_{h\theta}) = a_0 h \theta \partial_x^k + a_1 h' \theta \partial_x^{k-1} \quad \text{and} \quad \Upsilon_{11\frac{1}{2}}(v_{h\theta}) = b_0 h \partial_\theta \partial_x^k + b_1 h' \partial_\theta \partial_x^{k-1}.$$

Let us now solve the equations (3.25). We obtain  $\lambda = \mu$  and  $\Upsilon_{\lambda,\lambda}(X_F) = F'$ . This completes the proof of part 1).

2) Consider a non trivial odd 1-cocycle  $\Upsilon_1$  for  $\mathfrak{osp}(1|2)$  in  $\mathfrak{D}_{\lambda,\mu}$  and its decomposition  $\Upsilon_1 = \Upsilon_{010} + \Upsilon_{01\frac{1}{2}} + \Upsilon_{100} + \Upsilon_{10\frac{1}{2}}$ , where

$$\begin{cases} \Upsilon_{010} & : \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\ \Upsilon_{01\frac{1}{2}} & : \mathfrak{sl}(2) \rightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu}, \\ \Upsilon_{100} & : \mathfrak{h} \rightarrow \mathcal{D}_{\lambda,\mu}, \\ \Upsilon_{10\frac{1}{2}} & : \mathfrak{h} \rightarrow \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}. \end{cases} \quad (3.26)$$

We must have  $(\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2})$  with  $k \in \mathbb{N} \setminus \{0\}$ . Moreover  $\Upsilon_1 = \Upsilon_{000} + \Upsilon_{00\frac{1}{2}} + \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}$  is a 1-cocycle for  $\mathcal{K}(1)$  if and only if

$$\begin{cases} \Upsilon_{010} & = \alpha_1 C_k + \alpha_2 \tilde{C}_k \\ \Upsilon_{01\frac{1}{2}} & = \beta_1 C_{k-1} + \beta_2 \tilde{C}_{k-1} \\ \Upsilon''[X_g, X_{\theta h}] & = \mathfrak{L}_{X_g}^{\lambda,\mu} \Upsilon''(X_{\theta h}) - \mathfrak{L}_{X_{\theta h}}^{\lambda,\mu} \Upsilon'(X_g), \\ \Upsilon'[X_{\theta h_1}, X_{\theta h_2}] & = \mathfrak{L}_{X_{\theta h_1}}^{\lambda,\mu} \Upsilon''(X_{\theta h_2}) + \mathfrak{L}_{X_{\theta h_2}}^{\lambda,\mu} \Upsilon''(X_{\theta h_1}), \end{cases} \quad (3.27)$$

where  $\Upsilon' = \Upsilon_{010} + \Upsilon_{01\frac{1}{2}}$  and  $\Upsilon'' = \Upsilon_{100} + \Upsilon_{10\frac{1}{2}}$ .

As above, we then can write

$$\Upsilon_{100}(X_{h\theta}) = a_0 h \theta \partial_x^k + a_1 h' \theta \partial_x^{k-1} \quad \text{and} \quad \Upsilon_{10\frac{1}{2}}(v_{h\theta}) = b_0 h \partial_\theta \partial_x^k + b_1 h' \partial_\theta \partial_x^{k-1}.$$

According to Lemma 3.3, the map  $\Upsilon_1$  is a non trivial 1-cocycle if and only if at least one of the maps  $\Upsilon_{010}$  and  $\Upsilon_{01\frac{1}{2}}$  is a non trivial 1-cocycle for  $\mathfrak{sl}(2)$ , that means  $(\alpha_1, \alpha_2, \beta_1, \beta_2) \neq (0, 0, 0, 0)$ . Let us determine the linear maps  $\Upsilon_{100}$  and  $\Upsilon_{10\frac{1}{2}}$ . Up to factor, we get:

$$\Upsilon_1 = \alpha_1 \Upsilon_{\frac{1-k}{2}, \frac{k}{2}} + \alpha_2 \tilde{\Upsilon}_{\frac{1-k}{2}, \frac{k}{2}} + a_0 \delta(2\theta \partial_x^k).$$

Thus, the cohomology classes of  $\Upsilon_{\frac{1-k}{2}, \frac{k}{2}}$  and  $\tilde{\Upsilon}_{\frac{1-k}{2}, \frac{k}{2}}$  generate  $H_1^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{1-k}{2}, \frac{k}{2}})$ . The proof is now complete.

## References

- [1] I. Basdouri, M. Ben Ammar, N. Ben Fraj, M. Boujelbene and K. Kammoun *Cohomology of the Lie Superalgebra of Contact Vector Fields on  $\mathbb{R}^{1|1}$  and Deformations of the Superspace of Symbols*, math.RT/0702645.
- [2] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, Plenum Publ. New York, 1986.
- [3] H. Gargoubi, *Sur la géométrie de l'espace des opérateurs différentiels lineaires sur  $\mathbb{R}$* , Bull. Soc. Roy. Sci. Liège. Vol. 69, 1, 2000, 2147.

- [4] H. Gargoubi, N. Mellouli and V. Ovsienko *Differential Operators on Supercircle: Conformally Equivariant Quantization and Symbol Calculus*, Letters in Mathematical Physics (2007) **79**: 5165.
- [5] P. B. A. Lecomte, *On the cohomology of  $\mathfrak{sl}(n+1; \mathbb{R})$  acting on differential operators and  $\mathfrak{sl}(n+1; \mathbb{R})$ -equivariant symbols*, Indag. Math. NS. 11 (1), (2000), 95–114.
- [6] A. Nijenuis, R. W. Richardson Jr., *Deformations of homomorphisms of Lie groups and Lie algebras*, Bull. Amer. Math. Soc. **73** (1967), 175–179.